Generalized functions - Dirac delta function

Delta function as a sequence of functions

Define A function $f(x)$ is called a good function if the following conditions are fulfilled :

- The function and all its derivatives $f^{(n)}(x)$ for $n = 0, 1, 2, \ldots$ exist everywhere

- $|f^{(n)}(x)|$ vanishes faster than any power of $1/|x|$ as $|x| \to \infty$

Now consider the following sequence of good functions :

$$\{\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x), \ldots\} \tag{1}$$

such that the limit of the Riemann integral exists for each of the functions in the sequence :

$$\int_{-\infty}^{\infty} \alpha_n(x) g(x) dx < \infty \text{ for all } n \tag{2}$$

for every good function $g(x)$. Then a generalized function $\chi(x)$ is defined as the limit of the sequence of these functions. Further, we can define the integral of a generalized function $\chi(x)$ with a good function $g(x)$ as:

$$\int_{-\infty}^{\infty} \chi(x) g(x) dx \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} \alpha_n(x) g(x) dx \tag{3}$$

As an example we have :

$$f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \quad \{n = 0, 1, 2, \ldots\} \tag{4}$$

defines a generalized function $\delta(x)$ Such that :

$$\int_{-\infty}^{\infty} \delta(x) g(x) dx \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x) dx \tag{5}$$

The above generalized function defines what is called the Dirac $\delta-$function. We can in fact do this integral explicitly :

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x) dx \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} g(x) dx \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} g(\frac{y}{n}) dy \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} [g(0) + \left(\frac{y}{n}\right) g^{(1)}(0) + \frac{1}{2!}\left(\frac{y}{n}\right)^2 g^{(2)}(0) + \ldots] dy
\end{aligned} \tag{6}$$

where we have Taylor expanded $g(y/n)$ since it is a good function. More over all the derivatives $g^{(n)}(0) < \infty$. Hence all the integrals in the series are finite term wise. Thus all the terms apart from the first term is zero in the $n \to \infty$ limit :

$$g(0)\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-y^2} + \lim_{n\to\infty}[\frac{g^{(1)}}{n\sqrt{\pi}}\int_{-\infty}^{\infty} ye^{-y^2}dy + \frac{g^{(2)}}{2!n^2\sqrt{\pi}}\int_{-\infty}^{\infty} y^2e^{-y^2}dy + \ldots] \tag{7}$$

using the fact that :

$$\int_{-\infty}^{\infty} e^{-y^2}dy = \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{8}$$

we have the above equal to $g(0)$. $\therefore$ This is the defining equation for the Dirac δ-function :

$$g(0) = \int_{-\infty}^{\infty} g(x)\delta(x)dx \tag{9}$$

Or more generally under translation :

$$\begin{aligned} g(a) &= \int_{-\infty}^{\infty} g(x+a)\delta(x)dx, \quad x+a \to x' \\ &= \int_{-\infty}^{\infty} g(x')\delta(x'-a)dx' \end{aligned} \tag{10}$$

Hence the integral with the delta function picks the value of the function at $x' - a = 0$, i.e. when the argument of the delta function becomes zero. If we plug in $g(x) = 1$ for all x, we obtain :

$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1 \tag{11}$$

A much more vague definiton is :

$$\begin{aligned} \delta(x) &= 0, \quad x \neq 0 \\ &= \infty, \quad x = 0 \end{aligned} \tag{12}$$

There are obviously other sequences that can be used to represent the Dirac $\delta-$ function. In that sense the sequence, say $\delta_n(x)$ chosen which go to $\delta(x)$ in the limit $n \to \infty$ is not unique. Let us look at a few other examples :

$$\begin{aligned} \delta_n(x) &= 0, \quad x < -\frac{1}{2n} \\ &= n, \quad -\frac{1}{2n} < x < \frac{1}{2n} \\ &= 0, \quad x > \frac{1}{2n} \end{aligned} \tag{13}$$

$$
\begin{aligned}
\delta_n(x) &= \frac{n}{\pi}\frac{1}{(1+n^2x^2)} \\
\delta_n(x) &= \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ixt}dt \\
\delta_n(x) &= \frac{1}{2\pi}\frac{\sin[(n+\frac{1}{2})x]}{\sin\frac{x}{2}}
\end{aligned}
\tag{14}
$$

Properties

- Even function : $\delta(-x) = \delta(x)$. Check that :

$$
f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx
\tag{15}
$$

Again :

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\delta(-x)dx, \quad x = -x' \\
&= \int_{\infty}^{-\infty} (-dx')f(-x')\delta(x') \\
&= \int_{-\infty}^{\infty} f(-x')\delta(x')dx' = f(0)
\end{aligned}
\tag{16}
$$

Hence $\delta(-x) = \delta(x)$.

- $\delta(ax) = \frac{1}{|a|}\delta(x)$ where $a \in \mathcal{R}$. This is proved in parts. Firstly consider the case $a > 0$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\delta(ax)dx \quad ax = y \\
&= \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)\frac{dy}{a} \\
&= \frac{f(0)}{a}, \qquad \text{note that for } a > 0,\ a = |a| \text{ hence} \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)\delta(x)dx
\end{aligned}
\tag{17}
$$

again if $a < 0$ we have :

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\delta(ax)dx \quad ax = y \\
&= \int_{\infty}^{-\infty} f\left(\frac{y}{a}\right)\frac{dy}{a} \quad \text{note that for } a<0,\ a=-|a|,\ \text{ hence} \\
&= \int_{\infty}^{-\infty} f\left(\frac{y}{a}\right)\frac{dy}{-|a|} \\
&= \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\frac{dy}{|a|} \\
&= \frac{f(0)}{|a|} \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)\delta(x)dx
\end{aligned} \tag{18}$$

comparing we have :

$$\delta(ax) = \frac{1}{|a|}\delta(x) \quad a \in \mathcal{R} \tag{19}$$

- If a function $g(x)$ has simple (multiplicity one) zeroes at the points $x = a_i$, then :

$$\delta(g(x)) = \sum_{a_i, g(a_i)=0} \frac{\delta(x-a_i)}{|g'(a_i)|} \tag{20}$$

This can be rigorously done using the representation of the delta function as a limit of a sequence, but we will do it in a less rigorous route. If a_i is a zero of the function $g(x)$, we can perform the Taylor series of the function about that point :

$$g(x) = g(a_i) + (x-a_i)g'(a_i) + \frac{(x-a_i)^2}{2!}g''(a_i) + \ldots \tag{21}$$

Or for very small interval $x - a_i = \epsilon$, we have, noting $g(a_i) = 0$

$$g(x) \sim (x-a_i)g'(a_i) \tag{22}$$

Note here $g'(a_i)$ are assumed to be non zero. Now note that the integral :

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))dx \tag{23}$$

can be split along the points where where $g(x) = 0$ and with the chosen labeled ordering $a_1 < a_2 < a_3 < \ldots$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\delta(g(x))dx \\
&= \int_{-\infty}^{a_1-\epsilon} f(x)\delta(g(x))dx + \int_{a_1-\epsilon}^{a_1+\epsilon} f(x)\delta(g(x))dx \\
&+ \int_{a_1+\epsilon}^{a_2-\epsilon} f(x)\delta(g(x))dx + \int_{a_2-\epsilon}^{a_2+\epsilon} f(x)\delta(g(x))dx + \ldots
\end{aligned} \tag{24}$$

looking at each terms in this we see that the argument of the $\delta-$ function becomes zero only over sections of the integral of the form :

$$\int_{a_i-\epsilon}^{a_i+\epsilon} f(x)\delta(g(x))dx \tag{25}$$

since $g(a_i) = 0$ and $a_i - \epsilon < a_i < a_i + \epsilon$. For all the other parts of the integral the argument of the delta function never becomes zero as $g(x) \neq 0$ between those intervals. Hence those parts automatically dont contribute since the $\delta-$ function is always zero there. Thus we have the only contributions to the integral:

$$\sum_{a_i} \int_{a_i-\epsilon}^{a_i+\epsilon} f(x)\delta(g(x))dx \tag{26}$$

As we have already argued that for small ϵ, around the points a_i we have : $g(x) = (x - a_i)g'(a_i)$. Plugging it in :

$$\begin{aligned}
&\sum_{a_i} \int_{a_i-\epsilon}^{a_i+\epsilon} f(x)\delta((x - a_i)g'(a_i))dx \\
&= \sum_{a_i} \int_{a_i-\epsilon}^{a_i+\epsilon} f(x)\frac{\delta(x - a_i)}{|g'(a_i)|}dx \\
&= \sum_{a_i} \frac{f(a_i)}{|g'(a_i)|} \\
&= \int_{-\infty}^{\infty} f(x)\sum_{a_i} \frac{\delta(x - a_i)}{|g'(a_i)|}
\end{aligned} \tag{27}$$

Hence proving the identity.

- One can define the action of the derivative on the delta function, which can be understood under the integration. Again one can make this more rigorous using sequence of functions, but we will be the shorter less rigorous route.

$$
\begin{aligned}
&\int_{-\infty} f(x)\delta'(x-a)dx \\
&= f(x)\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x-a)dx, \text{ Integration by parts} \\
&= 0 - f'(a)
\end{aligned} \tag{28}
$$

- Integration can also be defined. Let us make this part more rigorous. Define a sequence that goes to the $\delta-$function in the limit $n \to \infty$

$$
\delta_n(x) = \frac{n}{2\cosh^2 nx} \tag{29}
$$

such that :

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)dx \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\frac{n}{2\cosh^2 nx}dx, \quad nx = y \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} f\left(\frac{y}{n}\right)\frac{dy}{2\cosh^2 y} \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} [f(0) + \frac{y}{n}f'(0) + \frac{1}{2!}\left(\frac{y}{n}\right)^2 f''(0) + \ldots]\frac{dy}{2\cosh^2 y}, \quad \text{taking the limit} \\
&= f(0)\int_{-\infty}^{\infty} \frac{dy}{2\cosh^2 y} \\
&= f(0)\int_{-\infty}^{\infty} \frac{dy}{2(\frac{e^y+e^{-y}}{2})^2} \\
&= f(0)\int_{0}^{\infty} \frac{4e^{2y}dy}{(e^{2y}+1)^2}, \quad e^{2y} = t \\
&= f(0)\int_{1}^{\infty} \frac{2dt}{(t+1)^2} = f(0)
\end{aligned} \tag{30}
$$

Using this sequence we will now check the integral of the delta function :

$$\begin{aligned}
\int_{-\infty}^{x} \delta_n(x')dx' &= \int_{-\infty}^{x} \frac{n}{2\cosh^2 nx'}dx' \\
&= \int_{-\infty}^{x} \frac{2ne^{2nx'}dx'}{(e^{2nx'}+1)^2}, \quad e^{2nx'} = t \\
&= \int_{0}^{e^{2nx}} \frac{dt}{(t+1)^2} = -\frac{1}{e^{2nx}+1} + 1 \\
&= \frac{1}{2} + \frac{1}{2}\tanh(nx)
\end{aligned} \tag{31}$$

Now we are in a position to take the limit :

$$\begin{aligned}
\lim_{n\to\infty} 1 - \frac{1}{e^{2nx}+1} \quad & \to 1 - 0 = 1, \quad x > 0 \\
& \to 1 - 1 = 0, \quad x < 0 \\
& \to 1 - \frac{1}{1+1} = \frac{1}{2}, \quad x = 0
\end{aligned} \tag{32}$$

The $n \to \infty$ limit of this function is called the Heaviside step function and is written with notation $\Theta(x)$. Obviously now in the limit $n \to \infty$, the integral of the delta function is defined, as $\delta_n \to \delta(x)$:

$$\int_{-\infty}^{x} \frac{d\Theta(x')}{dx'}dx' = \Theta(x) - \Theta(-\infty) = \int_{-\infty}^{x} \delta(x')dx', \quad \Theta(-\infty) = 0 \tag{33}$$

which gives us the integral of the $\delta(x)$ and also that the derivative of the step function is the delta function. :

$$\begin{aligned}
& \lim_{n\to\infty} \frac{d}{dx}\left[\frac{1}{2} + \frac{1}{2}\tanh nx\right] = \lim_{n\to\infty} \frac{n}{2\cosh^2 nx} \\
& \Rightarrow \frac{d\Theta(x)}{dx} = \delta(x)
\end{aligned} \tag{34}$$

- Examples :

a) Let us evaluate the following integral :

$$\begin{aligned}
&\pi \int_{-\pi}^{\pi} e^{-|x|}\delta(\sin(\pi x))dx \\
&= \pi \int_{-\infty}^{\infty} e^{-|x|} \sum_{n=-\infty}^{\infty} \frac{\delta(x-n)}{|\pi \cos \pi n|}, \quad \text{using identity for } \delta(g(x)) \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-|x|}\delta(x-n) \\
&= \sum_{n=-\infty}^{\infty} e^{-|n|} \\
&= e^0 + 2[e^{-1} + e^{-2} + e^{-3} + \dots] \\
&= 1 + 2\frac{e^{-1}}{1-e^{-1}} \\
&= \frac{1+e^{-1}}{1-e^{-1}} = \frac{e+1}{e-1}
\end{aligned} \tag{35}$$

b) Let $\lfloor x \rfloor$ be the greatest integer function not exceeding x. Evaluate the integral :

$$\begin{aligned}
&\int_0^{\infty} \lfloor x \rfloor e^{-x} dx \\
&= \lfloor x \rfloor e^{-x} \Big|_0^{\infty} + \int_0^{\infty} \left(\frac{d}{dx}\lfloor x \rfloor\right) e^{-x} dx
\end{aligned} \tag{36}$$

where we have done integration by parts. The first term is zero after plugging in the limits. Now note that :

$$\lfloor x \rfloor = \sum_{n=1}^{\infty} \Theta(x-n) \tag{37}$$

which is to say it increases by 1 or jumps a step every time we cross an integer, thus being the sum of step functions at integer points. Using the identity for the derivative of $\Theta-$ function :

$$\frac{d}{dx}\sum_{n=1}^{\infty} \Theta(x-n) = \sum_{n=1}^{\infty} \delta(x-n) \tag{38}$$

plugging into the integral we obtain:

$$\begin{aligned}
&\int_0^{\infty} \sum_{n=1}^{\infty} \delta(x-n) e^{-x} dx \\
&\sum_{n=1}^{\infty} e^{-n} = \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1}
\end{aligned} \tag{39}$$

3 Dimensional Delta function and Electrostatics

We know that electric field due to a point charge sitting at origin is given by :

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\hat{r}}{r^2}, \quad r = |\vec{r}| \tag{40}$$

While the Potential for this case is given by :

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{41}$$

We can now perform the closed surface integral of $\vec{E}$ for a sphere of radius R centered around origin where the charge is situated, represented by S, and enclosing a volume V, :

$$\begin{aligned}\oint_S \vec{E}.d\vec{S} &= \int_0^{\pi}\int_0^{2\pi}\frac{1}{4\pi\epsilon_0}\frac{q\hat{r}}{R^2}.\hat{r}R^2\sin\theta\,d\theta\,d\phi, \quad \text{here } d\vec{S} = \hat{r}R^2\sin\theta\,d\theta\,d\phi \quad \text{and } r = R \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}4\pi R^2 = \frac{q}{\epsilon_0}\end{aligned} \tag{42}$$

While on the other hand using Gauss's divergence theorem we have:

$$\begin{aligned}&\oint_S \vec{E}.d\vec{S} = \int_V \vec{\nabla}.\vec{E}\,dV = -\int_V \nabla^2\Phi\,dV, \quad \vec{E} = -\vec{\nabla}\Phi \\ &\Rightarrow -\int_V \nabla^2\Phi\,dV = \frac{q}{\epsilon_0} \\ &\Rightarrow -\int_V \nabla^2\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)dV = \frac{q}{\epsilon_0} \\ &\Rightarrow \int_V \nabla^2\left(\frac{1}{r}\right)dV = -4\pi\end{aligned} \tag{43}$$

But the above volume integral yields 4π for any arbitrary radius $R > 0$ about origin, which means the contribution can only come from a single point inside the domain of integration at origin. This can only be true if :

$$\nabla^2\frac{1}{r} = -4\pi\delta(x)\delta(y)\delta(z) \tag{44}$$

as the integration of the above about any arbitrary radius containing the origin yields -4π. In general if the origin is shifted to any point $\vec{r'}$, then it is straight forward that, the the above analysis is carried forward, and we get :

$$\nabla^2\frac{1}{|\vec{r}-\vec{r'}|} = -4\pi\delta(x-x')\delta(y-y')\delta(z-z') \quad (0,0,0) \text{ shifted to } (x',y',z') \tag{45}$$

Thus we have the three dimensional generalization of the Dirac delta function which is written with notation $\delta^3(\vec{r}-\vec{r'})$. Moreover it is note worthy, that we can rewrite the above as:

$$\begin{aligned}
&\vec{\nabla}\left(\vec{\nabla}\frac{1}{|\vec{r}-\vec{r'}|}\right) = -4\pi\delta^3(\vec{r}-\vec{r'}) \\
&\Rightarrow \vec{\nabla}\left(\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\right) = 4\pi\delta^3(\vec{r}-\vec{r'})
\end{aligned} \tag{46}$$

As a consequence to electrostatics we immediately realize that the charge distribution due to point charges is written using $\delta^3(\vec{r}-\vec{r'})$:

$$\nabla^2\frac{q}{4\pi\epsilon_0|\vec{r}-\vec{r'}|} = -\frac{q\delta^3(\vec{r}-\vec{r'})}{\epsilon_0} \equiv -\frac{\rho(\vec{r})_{\text{point}}}{\epsilon_0} \tag{47}$$

The three dimensional delta function can also be written in different coordinate systems :

- Spherical Polar coordinates :

$$\delta^3(\vec{r}-\vec{r'}) = \frac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\phi-\phi') \tag{48}$$

We can readily check this :

$$\begin{aligned}
&\int_0^\infty\int_0^\pi\int_0^{2\pi} f(r,\theta,\phi)\delta^3(\vec{r}-\vec{r'})r^2dr\sin\theta d\theta d\phi \\
&= \int_0^\infty\int_0^\pi\int_0^{2\pi} f(r,\theta,\phi)\frac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')r^2dr\sin\theta d\theta d\phi \\
&= \int_0^\infty\int_0^\pi\int_0^{2\pi} f(r,\theta,\phi)\frac{1}{r^2}\delta(r-r')\frac{\delta(\theta-\theta')}{|\sin\theta'|}\delta(\phi-\phi')r^2dr\sin\theta d\theta d\phi \\
&= f(r',\theta',\phi')\frac{\sin\theta'}{|\sin\theta'|} = f(r',\theta',\phi') \quad ,\sin\theta' = |\sin\theta'| \text{ since } \sin\theta' > 0 \text{ for } 0 < \theta' < \pi
\end{aligned} \tag{49}$$

- Cylindrical coordinates :

$$\delta^3(\vec{r}-\vec{r'}) = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \tag{50}$$

again we can check :

$$\begin{aligned}
&\int_0^\infty\int_0^{2\pi}\int_{-\infty}^\infty f(\rho,\phi,z)\delta^3(\vec{r}-\vec{r'})\rho\, d\rho d\phi dz \\
&= \int_0^\infty\int_0^{2\pi}\int_{-\infty}^\infty f(\rho,\phi,z)\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')\rho\, d\rho d\phi dz \\
&= f(\rho',\phi',z')
\end{aligned} \tag{51}$$

- Examples : **a)**Evaluate the integral :

$$\int_{\text{all space}} (r^2 + \vec{r}.\vec{a} + a^2)\delta^3(\vec{r} - \vec{a})dx\,dy\,dz \tag{52}$$

we can write $\delta^3(\vec{r} - \vec{a}) = \delta(x - a_1)\delta(y - a_2)\delta(z - a_3)$, where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Plugging it in :

$$\begin{aligned}&\int_{\text{all space}} (x^2 + y^2 + z^2 + xa_1 + ya_2 + za_3 + a_1^2 + a_2^2 + a_3^2)\delta(x - a_1)\delta(y - a_2)\delta(z - a_3)dx\,dy\,dz \\ &= 3(a_1^2 + a_2^2 + a_3^2) = 3\vec{a}.\vec{a}\end{aligned} \tag{53}$$

b) Evaluate the integral :

$$\int_V |\vec{r} - \vec{b}|^2\delta^3(5\vec{r})dx\,dy\,dz \tag{54}$$

where V is a cube of side 2 centered at origin and $\vec{b} = 4\hat{j} + 3\hat{k}$. Thus :

$$|\vec{r} - \vec{b}| = \sqrt{(x - 0)^2 + (y - 4)^2 + (z - 3)^2} \tag{55}$$

and

$$\begin{aligned}\delta^3(5\vec{r}) &= \delta(5x)\delta(5y)\delta(5z) \\ &= \frac{\delta(x)}{5}\frac{\delta(y)}{5}\frac{\delta(z)}{5}\end{aligned} \tag{56}$$

Plugging all this in the integral we have:

$$\begin{aligned}&\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz(x^2 + (y - 4)^2 + (z - 3)^2)\frac{\delta(x)}{5}\frac{\delta(y)}{5}\frac{\delta(z)}{5} \\ &= \frac{4^2 + 3^2}{5^3} = \frac{25}{125} = \frac{1}{5}\end{aligned} \tag{57}$$

c)Evaluate :

$$\int_V (r^4 + r^2(\vec{r}.\vec{c}) + c^4)\delta^3(\vec{r} - \vec{c})dx\,dy\,dz \tag{58}$$

where $\vec{c} = 5\hat{i} + 3\hat{j} + 2\hat{k}$, and V is a sphere of radius 6 about origin. In this case note that note first :

$$|\vec{c}| = \sqrt{25 + 9 + 4} = \sqrt{38} > 6 \tag{59}$$

We can write the volume integral in spherical polar coordinates:

$$\int_0^6 r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi (r^4 + r^2(\vec{r}.\vec{c}) + c^4)\frac{1}{r^2}\delta(r - \sqrt{38})\frac{\delta(\theta - \theta')}{|\sin\theta'|}\delta(\phi - \phi') = 0 \tag{60}$$

Where ϕ' and θ' is computed from the direction of $\vec{c}$ but is unimportant for the calculation since the delta function in r is used to see that the integral is 0 since $\sqrt{38}$ is not inside the range of integration over r.

d) Evaluate:

$$\int_V e^{-r}\left(\vec{\nabla}.\frac{\hat{r}}{r^2}\right) dx\, dy\, dz \tag{61}$$

where V is a sphere of radius R. This is easily evaluated noting :

$$\vec{\nabla}.\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r}) \tag{62}$$

Plugging it in and writing the volume integration in spherical polar coordinates :

$$\int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, e^{-r} 4\pi\delta^3(\vec{r}) = 4\pi \tag{63}$$